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Polynomial Shift States of a Chaotic Map

Dean J. Driebe[1] and Gonzalo E. Ordóñez[1]

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We present a piecewise-linear map of the unit interval in which the resolvent of the Frobenius-Perron operator, considered in a polynomial basis, has an essential singularity at the origin. Associated with the essential singularity are polynomial shift states, which are obtained from creation and annihilation operators in non-self-dual function spaces. Correlation functions of general polynomial observables have decay components that vanish in a finite time.

KEY WORDS: Frobenius-Perron operator; generalized spectral decomposition; finite correlation time; approach to equilibrium.

I. INTRODUCTION

A spectral decomposition of the Frobenius-Perron operator[1] solves the time evolution of a chaotic map on the level of the probability density. Decompositions that contain explicitly the decay rates and decay modes characterizing the approach to equilibrium have been obtained for a variety of one- and two-dimensional systems.[2-4] These generalized spectral decompositions are valid for smooth densities expandable in terms of polynomials. In this domain, for a class of one-dimensional piecewise-linear maps, the Frobenius-Perron operator has discrete eigenvalues corresponding to polynomial eigenstates. The dual states, which are eigenstates of the Koopman operator, are generalized functions.

Correlation functions for a class of observables in fully chaotic systems typically approach equilibrium as a sum of exponentially decaying contributions. A pure exponential decay contribution arises from a simple pole of the resolvent of the time evolution operator. If the resolvent has a multiple pole

[1] Center for Studies in Statistical Mechanics and Complex Systems, University of Texas, Austin, Texas 78712, and International Solvay Institutes for Physics and Chemistry, 1050 Brussels, Belgium.

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there will be associated modified exponential decay with factors polynomial in time. Such variety of time dependence is already observed in the class of one-dimensional piecewiselinear maps with the slope of all the branches having the same absolute value.[5, 6]

Here we examine a system belonging to this class with a new type of decomposition. The resolvent of the Frobenius–Perron operator, considered in a polynomial basis, has an essential singularity at the origin as well as simple poles. The part of the spectral decomposition associated with the essential singularity is in terms of a basis of shift states. The shift states are polynomials (of odd degree) so that correlation functions involving polynomial observables will contain in general both exponential decay contributions and contributions that vanish in finite times.

II. THE MAP

We consider the four-branch piecewise-linear map on the unit interval phase space given by

$$x_{t+1} = S_A(x_t) = \begin{cases} 4x_t & 0 \leqslant x_t < 1/4 \\ 2-4x_t & 1/4 \leqslant x_t < 1/2 \\ 3-4x_t & 1/2 \leqslant x_t < 3/4 \\ 4x_t - 3 & 3/4 \leqslant x_t \leqslant 1 \end{cases} \tag{1}$$

The map is shown in Fig. 1. Probability densities, $\rho(x, t)$, on the phase space evolve by application of the linear Frobenius–Perron operator, U_{A}, as $\rho(x, t+1) = U_{\mathrm{A}} \rho(x, t)$. The Frobenius–Perron operator for this map acts on a density as

$$U_{\mathrm{A}} \rho(x) = \frac{1}{4}\left[\rho\left(\frac{x}{4}\right) + \rho\left(\frac{2-x}{4}\right) + \rho\left(\frac{3-x}{4}\right) + \rho\left(\frac{3+x}{4}\right)\right] \tag{2}$$

The map preserves Lebesgue measure and the stationary (equilibrium) density is the uniform density, $\rho^{\mathrm{eq}}(x) = 1$.

We consider the spectral decomposition of U_{A} in a space spanned by polynomials. The action of U_{A} on a polynomial gives another polynomial of equal or lesser degree. The operator U_{A} is thus represented as a triangular matrix when acting on polynomials and the eigenvalues are found along the diagonal. Associated with polynomials of even-degree $2n$ we find eigenvalues of 4^{-2n}. The odd-degree polynomials are associated with the eigenvalue 0 of infinite multiplicity.

Because of the symmetry of the map[4] with respect to the midpoint of the unit interval, it is convenient to consider the reflection operator, R,

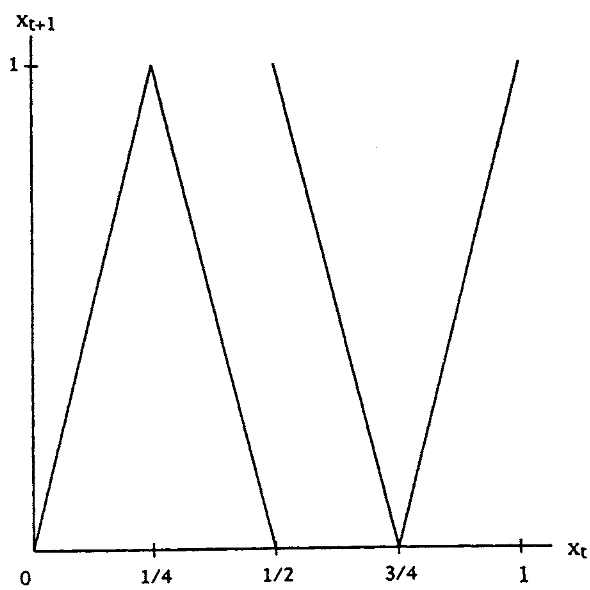

Fig. 1. The map S_A.

acting on functions in the phase space as $Rf(x) \equiv f(1-x)$. The Frobenius–Perron operator U_A commutes with R, i.e., $U_A R = R U_A$. Thus, U_A also commutes with the projection operators P_+ and P_- defined by

$$P_{\pm} \equiv \frac{1 \pm R}{2} \tag{3}$$

These operators project onto the even and odd parts, respectively, of a function with respect to the midpoint of the unit interval. The dynamics of the density thus decomposes into the independently evolving functional subspaces defined by the P_+ and P_- projections. We will thus consider the spectral decomposition of U_A separately in the two subspaces.

A. Decomposition in P_+

The operator U_A intertwines[4] with P_+ and the Frobenius–Perron operator of the 4-adic map, U_4, as

$$P_+ U_A = U_4 P_+ \tag{4}$$

where

$$U_4\rho(x)=\frac{1}{4}\left[\rho\left(\frac{x}{4}\right)+\rho\left(\frac{1+x}{4}\right)+\rho\left(\frac{2+x}{4}\right)+\rho\left(\frac{3+x}{4}\right)\right] \tag{5}$$

Since U_A commutes with P_+, the dynamics of U_A in this subspace is just like the 4-adic map, so this part of their spectral decomposition is identical. The generalized spectral decomposition of the 4-adic map is known.[2] The even-order right eigenpolynomials of U_A are thus the even-order Bernoulli polynomials, $B_{2n}(x)$, with associated eigenvalues of 4^{-2n}. These eigenvalues correspond to the location of simple poles of the resolvent of U_A considered in a polynomial basis. The duals, i.e., left eigenstates, are the eigenfunctionals, $\langle \tilde{B}_{2n}|$, defined by

$$\tilde{B}_{2n}(x)=\frac{(-1)^{2n-1}}{(2n)!}\left[\delta^{(2n-1)}(x-1)-\delta^{(2n-1)}(x)\right] \tag{6}$$

These functionals act on a density, $\rho(x)$, as

$$\langle \tilde{B}_{2n} \mid \rho\rangle=\int_0^1 dx\, \tilde{B}_{2n}(x)\, \rho(x) \tag{7}$$

The decomposition of U_A in P_+ thus has the explicit form

$$U_A P_+=\sum_{n=0}^{\infty}\frac{1}{4^{2n}}|B_{2n}\rangle\langle \tilde{B}_{2n}| \tag{8}$$

where we have also employed a Dirac-style notation for the right eigenpolynomials.

B. Decomposition in P_-

As noted above, the only eigenvalue of U_A associated with polynomials in the P_- subspace is zero. But $U_A P_- \neq 0$, which tells us that the eigenstates associated with the zero eigenvalue do not span P_-. In contrast, for the well-known two-branch tent map,[4] P_- is the null space of its Frobenius–Perron operator and any odd-symmetric polynomial is an eigenstate.

To determine whether there are distinct eigenstates associated with the zero eigenvalue of U_A we consider the nature of the singularity of its resolvent, i.e., $1/(z-U_A)$, at $z=0$. We write the resolvent in P_-, as spanned by the odd-degree Bernoulli polynomials and their duals, in terms of its matrix elements in this basis as

$$P_{-}\frac{1}{z-U_{A}}P_{-}=\sum_{m,m'=0}^{\infty}|B_{2m+1}\rangle\langle\tilde{B}_{2m+1}|\frac{1}{z-U_{A}}|B_{2m'+1}\rangle\langle\tilde{B}_{2m'+1}| \tag{9}$$

We then expand the resolvent in terms of the diagonal, U_0, and off-diagonal (strictly triangular) part, δU, of U_A to obtain for the matrix elements

$$\begin{aligned}\langle\tilde{B}_{2m+1}|\frac{1}{z-U_{A}}|B_{2m'+1}\rangle&=\sum_{n=0}^{\infty}\langle\tilde{B}_{2m+1}|\frac{1}{z-U_{0}}\left(\delta U\frac{1}{z-U_{0}}\right)^{n}|B_{2m'+1}\rangle\\&=\sum_{n=0}^{\infty}\langle\tilde{B}_{2m+1}|\frac{1}{z}\left(\delta U\frac{1}{z}\right)^{n}|B_{2m'+1}\rangle\end{aligned} \tag{10}$$

where we used that $P_{-}U_0P_{-}$ is the zero matrix. Since the off-diagonal matrix elements,

$$\langle\tilde{B}_{2m+1}|\,\delta U\,|B_{2m'+1}\rangle=\frac{(2m'+1)!\,B_{2m'-2m+1}(1/4)}{4^{2m}(2m+1)!\,(2m'-2m+1)!} \tag{11}$$

are nonvanishing (for $m'>m$), powers of $1/z$ up to $(1/z)^{m'-m+1}$ appear in (10). Hence, in (9) there appears an essential singularity at $z=0$. This means that U_A is not diagonalizable in this functional space but is only reducible to one Jordan block of infinite size, with zeroes on the diagonal.

When U_A acts on a polynomial in P_{-} of degree $2n+1$ it gives a polynomial of degree $2n-1$. Denoting a shift polynomial of degree $2n+1$ as $\phi_{2n+1}(x)$ we have

$$U_{A}\phi_{2n+1}(x)=\phi_{2n-1}(x) \tag{12}$$

for $n\geqslant 1$, where we have incorporated here any weight factors into the definition of the states. The first-degree polynomial associated with the zero eigenvalue is the only eigenstate of U_A in P_{-}. This state is unique (up to a constant factor) and is $\phi_1(x)=x-1/2$. We refer to this state as the vacuum state. The decomposition of U_A in P_{-} is formally thus

$$U_{A}P_{-}=\sum_{n=1}^{\infty}|\phi_{2n-1}\rangle\langle\tilde{\phi}_{2n+1}| \tag{13}$$

III. CONSTRUCTION OF POLYNOMIAL SHIFT STATES

We first note that any polynomial in P_-, say $p_{2n+1}(x)$, may be used to generate a set of shift states running down to the vacuum state as $\{p_{2n+1}, U_A p_{2n+1}, U_A^2 p_{2n+1},..., U_A^n p_{2n+1}\}$. Since $U_A U_A^\dagger = 1$ we may consider applying $U_A^\dagger$ successively to $p_{2n+1}(x)$ to generate shift states above $p_{2n+1}(x)$. But these states will not be higher-degree polynomials. The operator $U_A^\dagger$ acts on a function as $U_A^\dagger f(x) = f(S_A(x))$, where $S_A(x)$ is the rule (1) for the map.

There are several ways to construct complete families of polynomial shift states. Since the action of U_A reduces the degree of a shift polynomial by two, it motivates the consideration of families of shift polynomials that satisfy

$$\frac{d^2}{dx^2} \psi_{2n+1}(x) = v_{2n-1} \psi_{2n-1}(x) \tag{14a}$$

For states in this family U_A will act in general as a weighted shift,

$$U_A \psi_{2n+1}(x) = w_{2n-1} \psi_{2n-1}(x) \tag{14b}$$

The weight w_{2n-1} may be found by comparing the highest degree terms on the right hand sides of both (14a) and (14b),

$$w_{2n-1} = \frac{v_{2n-1}}{2 \cdot 4^{2n+1}} \tag{15}$$

The fact that (14a) and (14b) are consistent may be proven from the intertwining relation: $(d^2/dx^2)\, U_A = (1/4^2)\, U_A (d^2/dx^2)$. To construct ψ_{2n+1} from ψ_{2n-1} we invert (14a), i.e., integrate twice, and use (14b) to determine the constants of integration.

Suppose that the state ψ_{2n-1} is known and is expanded in terms of Bernoulli polynomials as

$$\psi_{2n-1}(x) = \sum_{j=1}^{n} b_{2j-1}^{(2n-1)} B_{2j-1}(x) \tag{16}$$

We want to find the expansion coefficients, $b_{2j-1}^{(2n+1)}$ of the next shift state, $\psi_{2n+1}(x)$. Using that[7] $(d^2/dx^2)\, B_{2j+1}(x) = 2j(2j+1)\, B_{2j-1}(x)$ we obtain, from (14a)

$$b_{2j+1}^{(2n+1)} = v_{2n-1} \frac{b_{2j-1}^{(2n-1)}}{2j(2j+1)}, \qquad j = 1,..., n \tag{17}$$

The coefficient, $b_1^{(2n+1)}$, of the vacuum component of $\psi_{2n+1}(x)$ is still undetermined. Since this component is annihilated by U_A and by the second derivative operator, we need to go one more step up; that is, consider $\psi_{2n+3}(x)$ in order to determine this coefficient. The state $\psi_{2n+3}(x)$ is obtained from (14a) by integrating $\psi_{2n+1}(x)$ twice as,

$$\psi_{2n+3}(x) = b_1^{(2n+3)} B_1(x) + v_{2n+1} \sum_{j=0}^{n} b_{2j+1}^{(2n+1)} \frac{B_{2j+3}(x)}{(2j+3)(2j+2)} \tag{18}$$

Using (14b) we extract the $B_1(x)$ component of $\psi_{2n+1}(x)$ from $\psi_{2n+3}(x)$ as

$$b_1^{(2n+1)} = \langle \tilde{B}_1 \mid \psi_{2n+1} \rangle = \frac{1}{w_{2n+1}} \langle \tilde{B}_1 | \, U_A \, | \psi_{2n+3} \rangle \tag{19}$$

Then from (18), using (17) and that $\langle \tilde{B}_1 | \, U_A \, | B_{2j+3} \rangle = B_{2j+3}(1/4)$ we can solve for $b_1^{(2n+1)}$ to obtain

$$b_1^{(2n+1)} = -\frac{2 \cdot 4^3 v_{2n-1}}{1 - 4^{-2n}} \sum_{j=1}^{n} \frac{b_{2j-1}^{(2n-1)} B_{2j+3}(1/4)}{(2j)(2j+1)(2j+3)(2j+2)} \tag{20}$$

The above procedure can be written in terms of a creation operator, U_C, which acts as $U_C \psi_{2n-1}(x) = \psi_{2n+1}(x)$. The creation operator is given by

$$U_C = \sum_{j=1}^{\infty} \frac{v_{2n-1}}{2j(2j+1)} \left[|B_{2j+1}\rangle - \frac{2 \cdot 4^3}{1 - 4^{-2n}} \frac{B_{2j+3}(1/4)}{(2j+3)(2j+2)} |B_1\rangle \right] \langle \tilde{B}_{2j+1} | \tag{21}$$

If we take $v_{2n-1} = (2n+1)(2n)$ then we will generate shift states that are monic polynomials. We start from the vacuum state and work our way up. Explicit forms of the first few monic polynomial shift states generated this way are

$$\begin{aligned}
\psi_1(x) &= x - \tfrac{1}{2} \\
\psi_3(x) &= x^3 - \tfrac{3}{2}x^3 + \tfrac{2}{3}x - \tfrac{1}{12} \\
\psi_5(x) &= x^5 - \tfrac{5}{2}x^4 + \tfrac{20}{9}x^3 - \tfrac{5}{6}x^2 + \tfrac{16}{135}x - \tfrac{1}{270} \\
\psi_7(x) &= x^7 - \tfrac{7}{2}x^6 + \tfrac{14}{3}x^5 - \tfrac{35}{12}x^4 + \tfrac{112}{135}x^3 - \tfrac{7}{90}x^2 - \tfrac{16}{8505}x + \tfrac{1}{68040} \\
\psi_9(x) &= x^9 - \tfrac{9}{2}x^8 + 8x^7 - 7x^6 + \tfrac{224}{75}x^5 - \tfrac{7}{15}x^4 - \tfrac{64}{2835}x^3 \\
&\quad + \tfrac{1}{1890}x^2 + \tfrac{7424}{3614625}x - \tfrac{29}{7229250}
\end{aligned} \tag{22}$$

A. Dual Shift States

The duals, $\tilde{\phi}_{2n+1}(x)$, of the generic shift polynomials obeying (12) satisfy $\langle \tilde{\phi}_{2n+1} \mid \phi_{2m+1} \rangle = \delta_{n,m}$. They can be obtained from successive applications of $U_{\rm A}^{\dagger}$ to $\tilde{\phi}_1(x)$ as $(U_{\rm A}^{\dagger})^n \tilde{\phi}_1(x) = \tilde{\phi}_{2n+1}(x)$. The dual of the vacuum state in terms of the duals of the Bernoulli polynomials and the expansion coefficients of the shift states is

$$\tilde{\phi}_1(x) = \frac{1}{b_1^{(1)}} \left\{ \tilde{B}_1(x) - \frac{b_1^{(3)}}{b_3^{(3)}} \tilde{B}_3(x) - \frac{1}{b_5^{(5)}} \left(b_1^{(5)} - \frac{b_3^{(5)} b_1^{(3)}}{b_3^{(3)}} \right) \tilde{B}_5(x) - \cdots \right\} \tag{23}$$

We note that $U_{\rm C}^{\dagger}$ acts as an annihilation operator on the dual states as $U_{\rm C}^{\dagger} \tilde{\phi}_{2n+1}(x) = \tilde{\phi}_{2n-1}(x)$.

For the families of states obeying (14) the higher-order duals can be expressed simply in terms of the vacuum dual state as

$$\langle \tilde{\psi}_{2n+1} | = \prod_{k=0}^{n-1} [v_{2k+1}]^{-1} \langle \tilde{\psi}_1 | \frac{d^{2n}}{dx^{2n}} \tag{24}$$

IV. CORRELATIONS AND POWER SPECTRA

The effect of the polynomial shift dynamics can be seen on correlation functions involving observables that are polynomials in x. The correlation between two observables, $A(x)$ and $B(x)$, is defined by

$$C_{BA}(t) = \lim_{T \to \infty} \frac{1}{T} \sum_{\tau=0}^{T-1} B(x_{t+\tau})\, A(x_\tau) = \int_0^1 dx\, B(x)\, U_{\rm A}^t A(x) \tag{25}$$

where we used the ergodicity of the map to replace the time average by a phase space average with respect to the equilibrium uniform density.

To see the effect of the shift dynamics we need to consider one of the observables to be at least a third degree polynomial. The simplest choice for illustration is to take $B(x) = x$ and $A(x) = \phi_{2n+1}(x)$. Then we obtain

$$C_{x\phi_{2n+1}}(t) = \langle x \mid U_{\rm A}^t \phi_{2n+1} \rangle = \sum_{m=0}^{n} a_{n-m}\, \delta_{t,m} \tag{26}$$

where $a_k \equiv \int_0^1 dx\, x \phi_{2k+1}(x)$. So the equilibrium value of the correlation, which here is zero, is attained for $t \geqslant n+1$.

The power spectral density, $S_{BA}(\omega)$, is given by the Fourier transform of the correlation function as $S_{BA}(\omega) \equiv \sum_{t=0}^{\infty} \cos \omega t C_{BA}(t)$. For the above correlation we thus obtain

$$S_{x\phi_{2n+1}}(\omega) = \sum_{t=0}^{\infty} \cos \omega t \sum_{m=0}^{n} a_{n-m} \delta_{t,m} = \sum_{m=0}^{n} a_{n-m} \cos m\omega \tag{27}$$

V. CONCLUDING REMARKS

We have presented a simple piecewise-linear map where correlation functions among polynomial observables have decay components vanishing in finite times. This behavior is a consequence of part of the spectral decomposition of the Frobenius–Perron operator of the map being in terms of polynomial shift states. This is also reflected in the analytic structure of the resolvent of the Frobenius-Perron operator. It has an essential singularity at $z = 0$.

Having a basis of shift states, like those obeying (12), motivates the consideration of coherent eigenstates[8] constructed from them as

$$\alpha_z(x) = \sum_{n=0}^{\infty} z^n \phi_{2n+1}(x) \tag{28}$$

But the radius of convergence of this series is zero, as can be determined from a careful consideration of the weight factors of properly normalized shift states. This is in contrast to the situation for the r-adic map, which has unweighted trigonometric shift states and regular coherent states.[2,3,8] It may be possible though to give a meaning here to (28) as a distribution.

An interesting feature of the analysis we have presented is the non-self-dual functional space setting for creation and annihilation operators. Instead of the pair of operators occuring in a standard quantum mechanical context, we have four operators: U_A, $U_A^\dagger$, U_C and $U_C^\dagger$. This is analogous to the second quantized formulation of the generalized spectral decomposition of the quantum mechanical Friedrichs model.[9]

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